



# Reconstruction of defects via multifrequency topological derivatives

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## Forward and inverse problems



Figure 1: Left and center: Motivation. Right: Geometrical setting of the problem.

**Forward problem:** We generate a time-harmonic wave at a frequency  $\omega_m$  from the emitter  $\mathbf{x}_j$ . The total wave  $u_{\Omega, \omega_m}^j$  solves

$$\begin{cases} \kappa \Delta u_{\Omega, \omega_m}^j + \rho \omega_m^2 u_{\Omega, \omega_m}^j = \delta_{\mathbf{x}_j}, & \text{in } D \setminus \Omega, \\ \partial_n u_{\Omega, \omega_m}^j = 0, & \text{on } \partial D \cup \partial \Omega. \end{cases} \quad (1)$$

**Inverse problem:** Measurements  $u_{meas}$  are taken at the receivers after exciting the system from the emitter  $\mathbf{x}_j$  ( $j = 1, \dots, n_{sc}$ ) at different frequencies  $\omega_m$  ( $m = 1, \dots, n_{freq}$ ). We want to find  $\Omega$  such that the solution  $u_{\Omega, \omega_m}^j$  of (1) satisfies

$$u_{\Omega, \omega_m}^j(\mathbf{x}_i) = u_{meas, \omega_m}^j(\mathbf{x}_i), \quad \begin{matrix} m = 1, \dots, n_{freq}, \\ j = 1, \dots, n_{sc}, \\ i = 1, \dots, n_{sc}, \quad i \neq j. \end{matrix} \quad (2)$$

We rewrite Problem (2) as:

Find  $\Omega$  minimizing

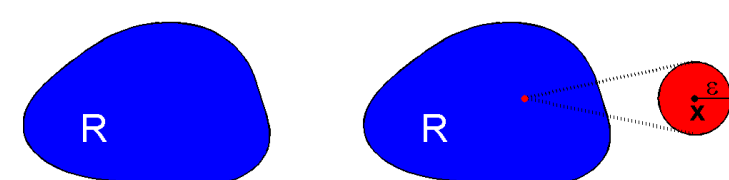
$$J(D \setminus \Omega) := \sum_{m=1}^{n_{freq}} \sum_{j=1}^{n_{sc}} \alpha_{m,j} J_{m,j}(D \setminus \Omega), \quad J_{m,j}(D \setminus \Omega) := \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{n_{sc}} \left( u_{\Omega, \omega_m}^j(\mathbf{x}_i) - u_{meas, \omega_m}^j(\mathbf{x}_i) \right)^2$$

where  $u_{\Omega, \omega_m}^j$  solves (1). The weights  $\alpha_{m,j} > 0$  account for the contribution of each emitter and each frequency. They will be selected after computing the topological derivative of each functional  $J_{m,j}$ .

## Topological derivative

The TD [1] of a shape functional  $J(\mathcal{R})$  at a point  $\mathbf{x} \in \mathcal{R}$  is

$$\mathcal{D}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) - J(\mathcal{R})}{\text{Vol}(B_\varepsilon(\mathbf{x}))}$$



- It is a scalar function of  $\mathbf{x}$  that measures the sensitivity to removing a ball around  $\mathbf{x}$ .
- We use it as an indicator function:

$$\mathcal{D}(\mathbf{x}) \ll 0 \implies \mathbf{x} \text{ belongs to a defect}$$

**Theorem 1** [3]. (a) The topological derivative of  $J_{m,j}$  is

$$\mathcal{D}_{m,j}(\mathbf{x}) = 2\kappa \nabla u_{\omega_m}^j(\mathbf{x}) \cdot \nabla p_{\omega_m}^j(\mathbf{x}) - \rho \omega_m^2 u_{\omega_m}^j(\mathbf{x}) p_{\omega_m}^j(\mathbf{x}), \quad \mathbf{x} \in D,$$

where  $u_{\omega_m}^j$  and  $p_{\omega_m}^j$  solve direct and adjoint problems in  $D$ :

$$\text{Direct problem: } \begin{cases} \kappa \Delta u_{\omega_m}^j + \rho \omega_m^2 u_{\omega_m}^j = \delta_{\mathbf{x}_j}, & \text{in } D, \\ \partial_n u_{\omega_m}^j = 0, & \text{on } \partial D. \end{cases}$$

$$\text{Adjoint problem: } \begin{cases} \kappa \Delta p_{\omega_m}^j + \rho \omega_m^2 p_{\omega_m}^j = \sum_{\substack{i=1 \\ i \neq j}}^{n_{sc}} \left( u_{meas, \omega_m}^j(\mathbf{x}_i) - u_{\omega_m}^j(\mathbf{x}_i) \right) \delta_{\mathbf{x}_i}, & \text{in } D, \\ \partial_n p_{\omega_m}^j = 0, & \text{on } \partial D. \end{cases}$$

$$(b) \text{ The topological derivative of } J \text{ is } \mathcal{D}(\mathbf{x}) = \sum_{m=1}^{n_{freq}} \sum_{j=1}^{n_{sc}} \alpha_{m,j} \mathcal{D}_{m,j}(\mathbf{x}).$$

**Remark:** To compute the TD, we only have to solve the direct and adjoint problems. When  $D$  is either a circle or a rectangle, both solutions can be obtained using an analytical eigenfunction expansion. Otherwise, we use the FEM.

**Selection of the weights:** We make all the TDs equally important by defining

$$\alpha_{m,j} := \frac{1}{|\min_{\mathbf{y} \in D} \mathcal{D}_{m,j}(\mathbf{y})|}.$$

References:

- [1] J. Sokolowski and A. Zochowski (1999). On the topological derivative in shape optimization. *SIAM J. Control Optim.* **37**, 1251–1272.
- [2] J.F. Funes (2014). Defect detection in structures by topological sensitivity techniques. Master thesis, Universidad Politécnica de Madrid. In Spanish.
- [3] J.F. Funes, J.M. Perales, M.L. Rapún and J.M. Vega (2015). Defect detection from multifrequency limited data via topological sensitivity computations. Submitted.

## Numerical examples: full aperture

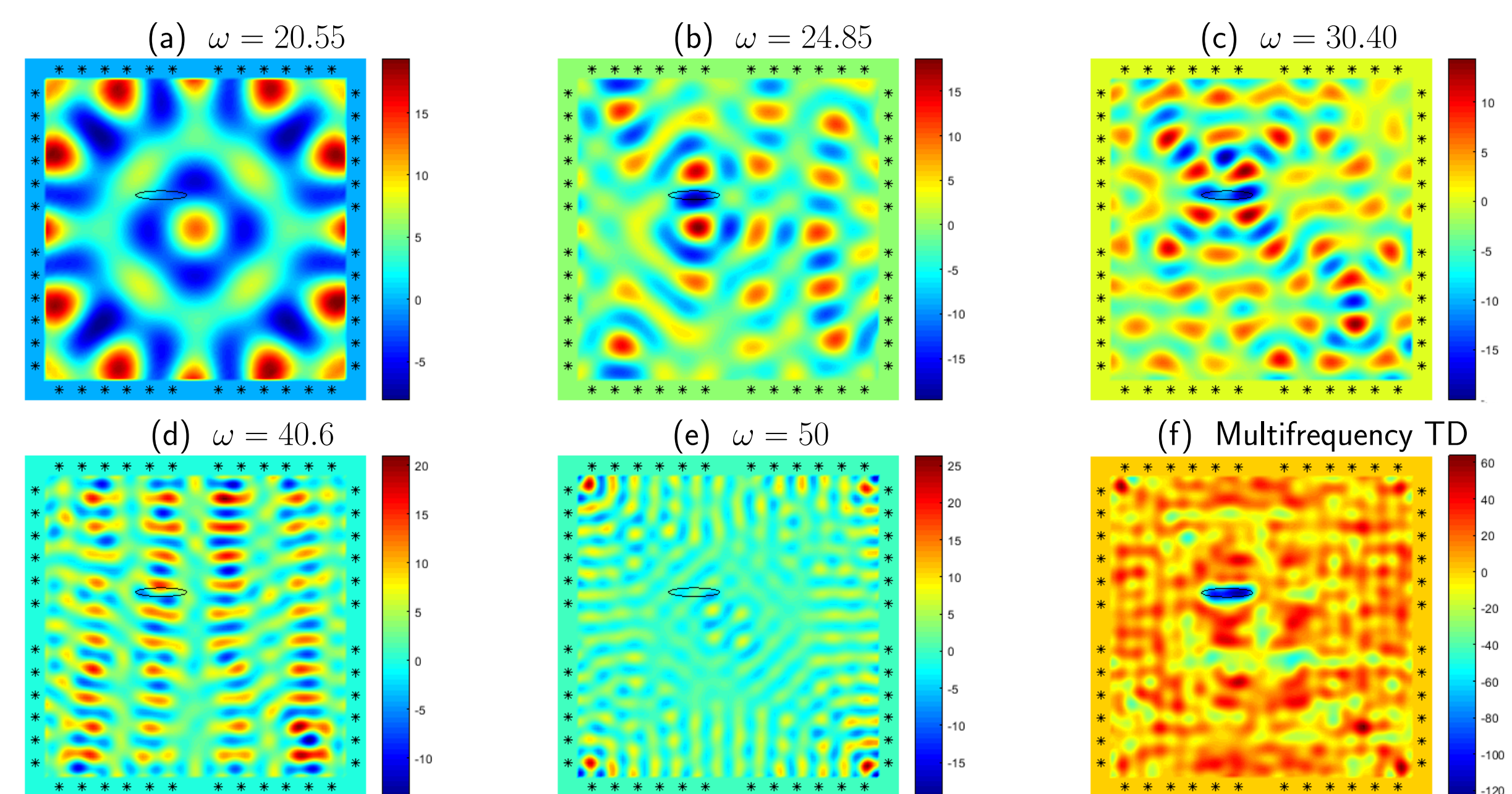


Figure 2: (a-e) TD for different frequencies  $\omega$ , selected to avoid eigenfrequencies of the Laplacian in  $D$ . The 48 emitters/receivers are located at " $\mathbf{x}_j$ ". A wave is emitted from each of them and measured at the remaining 47 ones. Dark blue colors indicate possible locations of the scatterers: the reconstructions are very poor. (f) Multifrequency TD combining 14 frequencies in  $[15,60]$ . The reconstruction greatly improves.

## Numerical examples: very limited aperture

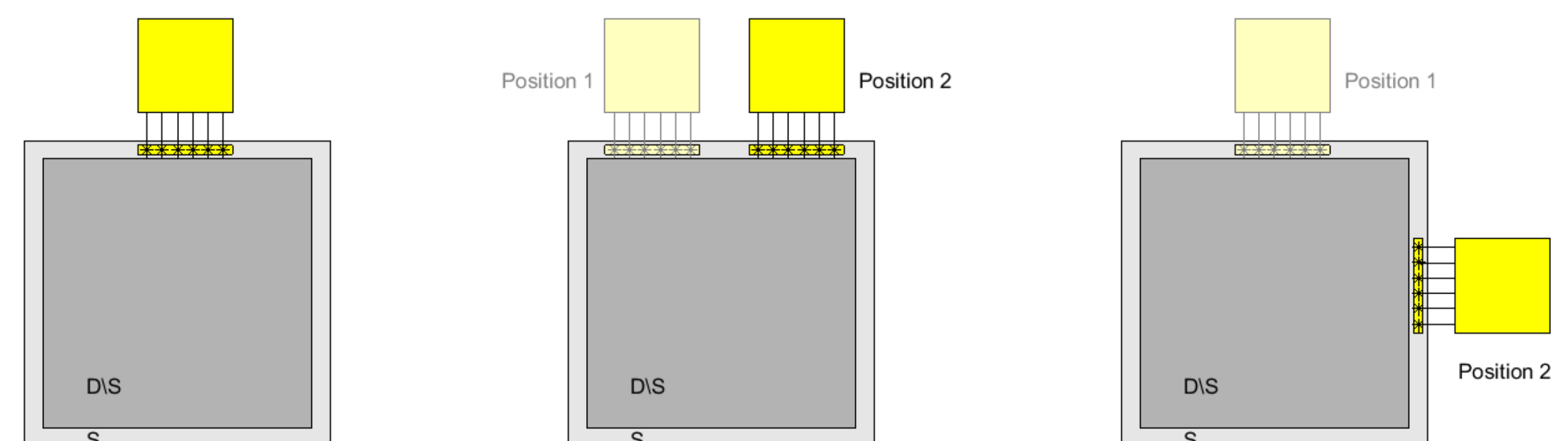


Figure 3: A device with six aligned emitters/receivers. The device can be moved to a different position.

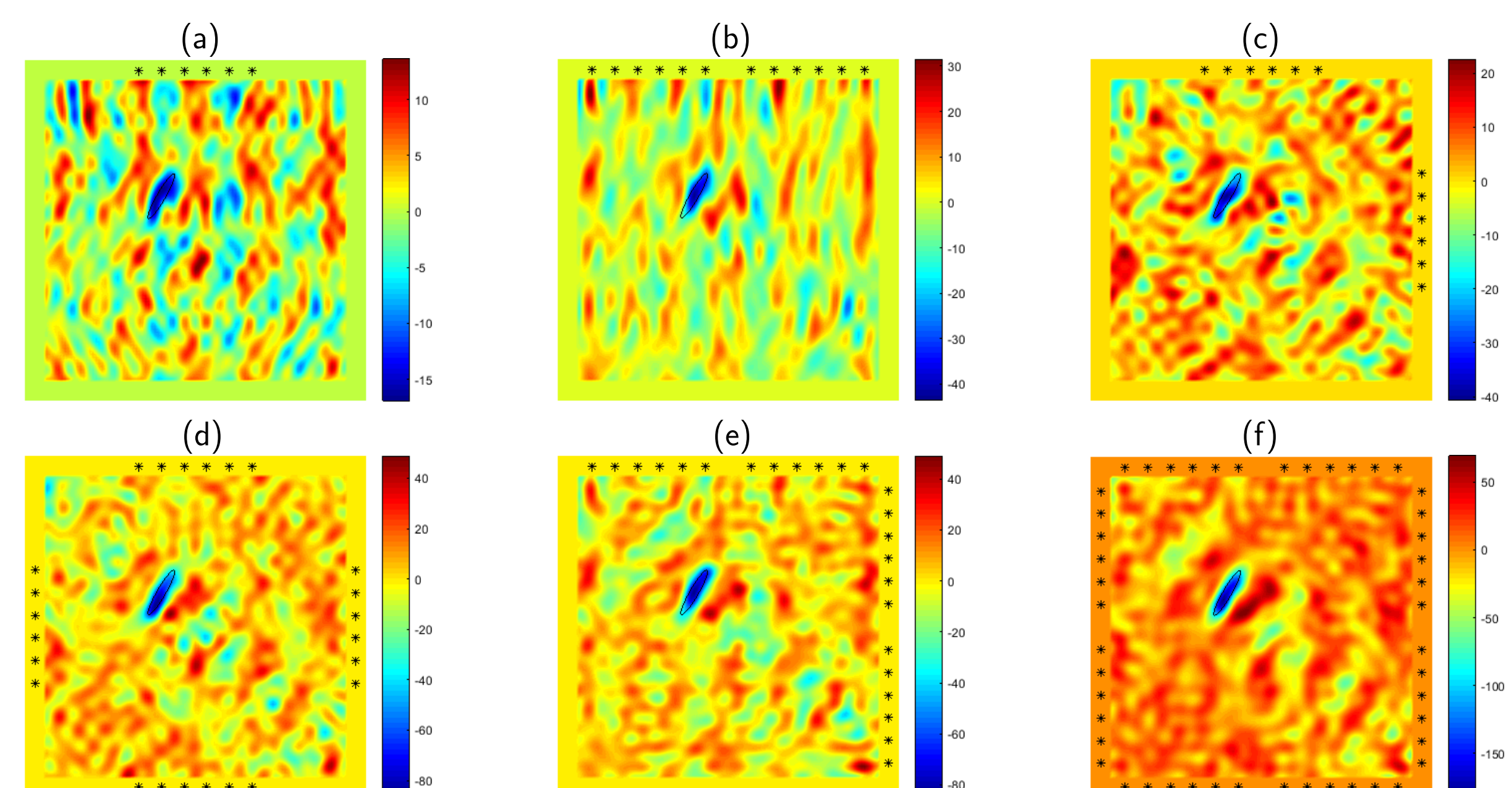


Figure 4: Multifrequency TD combining 14 frequencies in  $[15,60]$ . The device is located at different positions. (a) The TD detects the location, shape, size and orientation, but spurious regions appear. (b,c) By locating the device at two positions spurious regions fade. (d,e,f) Four or eight locations provide clear reconstructions, comparable to full aperture simulations (see Fig.2(f)).

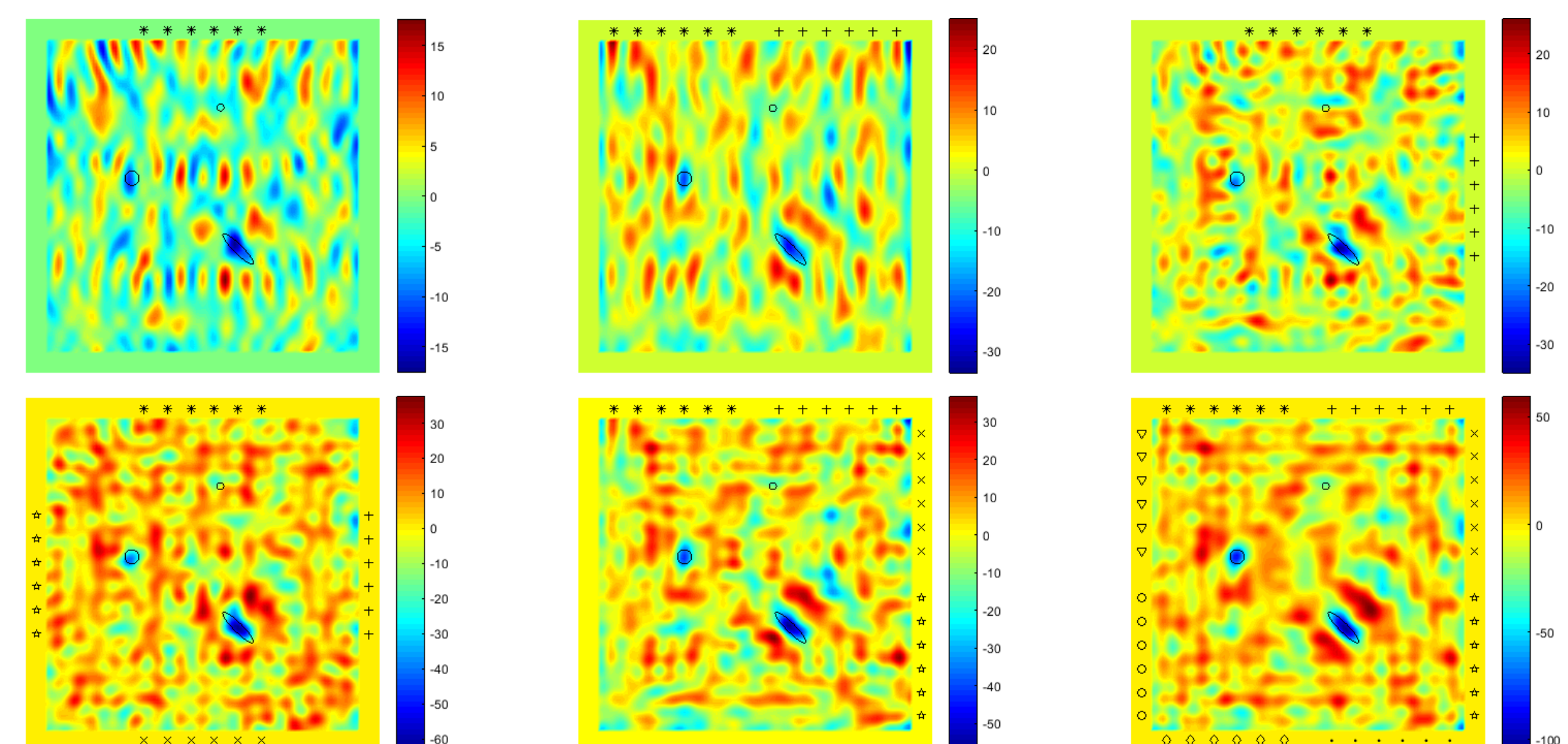


Figure 5: Counterpart of Figure 4 when three defects of different sizes are buried in  $D$ . The TD is more sensitive to the bigger defects that are identified with accuracy. The smaller one is not detected.

## Conclusions

- Very reasonable reconstructions at a low computational cost, even when a reduced number of aligned emitters and receivers is considered.
- Our method can be used to provide a good initial guess for more sophisticated iterative methods when very precise detection is desired.
- Further details and simulations in [2,3]. Contact: [marialuisa.rapun@upm.es](mailto:marialuisa.rapun@upm.es)